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Dirac Equation in (1 + 1)-Dimensional Curved Space-Time

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The Dirac equation in (1 + 1)-dimensional curved space-time is solved explicitly for the spatially flat Robertson–Walker space-time and the cigar metric considered by Witten.

1. INTRODUCTION

Of late, quantum theory in curved space-time including (1 + 1)-dimensional gravity has attracted considerable attention (Boulware, 1975*a,b*; Christensen and Fulling, 1977; Najmi and Hewill, 1984; Gegenberg *et al.*, 1988; Brown *et al.*, 1986; Mann *et al.*, 1991; see Birrell and Davies for general review). The (1 + 1)-dimensional analog of the Einstein equation for nonzero cosmological constant, namely $R - \Lambda = 0$, has been known for a long time. But it was only recently realized (Brown *et al.*, 1986, and references therein) that a nontrivial theory of gravity which has the field equation

$$R - \Lambda = 8\pi GT \tag{1.1}$$

can be obtained from a covariant action principle. In the case of the Dirac field the complete set of equations is

$$R - 8\pi T^{(\mu)} = 8\pi m\bar{\psi}\psi \tag{1.2}$$

where $T^{(\mu)}$ represents classical sources.

The Dirac field in two-dimensional curved space-time was considered by several authors (Gegenberg *et al.*, 1988; Brown *et al.*, 1986; Mann *et al.*,

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1991). Very recently, Mann *et al.* (1991) have shown that for a source particle theory, the metric is given by

$$ds^2 = \alpha(x)\,dt^2 - \frac{1}{\alpha(x)}\,dx^2 \tag{1.3}$$

where $\alpha(x)$ is of the form

$$\alpha(x) = \lambda x^2 + 2M|x| + c \tag{1.4}$$

Here we study the Dirac field in $(1+1)$-dimensional curved space-time for the cases

$$\text{(a)} \quad ds^2 = dt^2 - a^2(t)\,dx^2 \tag{1.5}$$

$$\text{(b)} \quad ds^2 = \tanh^2 x\,dt^2 - dx^2 \tag{1.6}$$

Case (a) involves spatially flat Robertson–Walker space-time and case (b) is, according to Witten (1991), the analytic continuation of the black hole (torsion-free) to a Lorentz signature.

It may be noted here that in $1+2$ dimensions also the Dirac equation can be studied in a similar manner, as the γ matrices still have 2×2 representations.

In Section 2 we briefly review the formalism of the Dirac equation in curved space-time using the dyad formalism. We also solve the Dirac equation for case (a) mentioned above. In Section 3 we solve the same for case (b). Section 4 has a conclusion and discussion.

2. DIRAC EQUATION IN CURVED SPACE-TIME

We introduce a locally inertial Minkowski frame where the γ matrices are usually defined. At each space-time point X, we introduce a local normal coordinate Y_X^a. In terms of Y_X^a, the metric at X is η_{ab}. In terms of a general coordinate system, the metric tensor will be

$$g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab} \tag{2.1}$$

where

$$e_\mu^a(x) = \left(\frac{\partial Y_X^a}{\partial x^\mu}\right)_{x=X}, \qquad a = 0, 1 \tag{2.2}$$

e_μ^a are the dyads which project vectors between the two frames. The γ matrices in curved space-time are written as

$$\gamma^\mu = e_a^\mu \gamma^a \tag{2.3}$$

where γ^a are the flat Minkowskian γ matrices which satisfy

$$\{\gamma^a, \gamma^b\} = 2n^{ab}, \qquad \frac{1}{4}[\gamma^a, \gamma^b] = \sigma^{ab} \tag{2.4}$$

so that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.5}$$

The Dirac equation in curved space-time can now be written as

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0 \tag{2.6}$$

where ∇_μ is the covariant derivative, defined to be

$$\nabla_\mu \equiv \partial_\mu + \Gamma_\mu \tag{2.7}$$

The Γ_μ are the spin connections given by

$$\Gamma_\mu = \frac{1}{2}\sigma^{bc} e^\nu_b e_{c\nu;\mu} \tag{2.8}$$

The Γ_μ satisfy the equation

$$[\Gamma_\mu, \gamma^\mu] = \frac{\partial \gamma^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho}\gamma^\rho \tag{2.9}$$

Here $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols for the metric under consideration

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\lambda\mu}[g_{\lambda\nu,\sigma} + g\lambda_{\sigma,\nu} - g_{\nu\sigma,\lambda}] \tag{2.10}$$

In $1 + 1$ dimensions, $\mu^\mu \Gamma_\mu$ simplifies greatly and using

$$\Gamma^\mu_{\nu\mu} = \frac{1}{\sqrt{-g}} \partial_\nu(\sqrt{-g}) \tag{2.11}$$

we can reduce it to

$$\mu^\mu \Gamma_\mu = \frac{1}{2}\gamma^a \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g}\, e^\mu_a) \tag{2.12}$$

Thus, the Dirac equation can now be written as

$$\left[i\gamma^a e^\mu_a \partial_\mu + \frac{i}{2}\gamma^a \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g}\, e^\mu_a) - m\right]\Psi = 0 \tag{2.13}$$

We now consider the metric[2]

$$ds^2 = dt^2 - a^2(t)\, dx^2 \tag{2.14}$$

[2]The Dirac equation in $3 + 1$ dimensions in a flat Robertson–Walker metric was considered by Barut and Duru (1987). See also Najmi and Hewill (1984). Here we have considered explicitly a case, namely $a(t) = a_0/t$, which was not considered before.

The dyads are given by

$$e_0^0 = 1, \qquad e_1^1 = \frac{1}{a}, \qquad e_0^1 = e_1^0 = 0 \tag{2.15}$$

Here the γ^a matrices are chosen as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.16}$$

From

$$\gamma^\mu = \gamma^a e_a^b \tag{2.17}$$

we get

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.18}$$

Equation (2.13) then reduces to

$$\left(\partial_t + \frac{\dot{a}}{2a} + \frac{ik}{a}\gamma^0\gamma^1 + im\gamma^a\right)\psi(t) = 0 \tag{2.19}$$

where we have taken

$$\begin{aligned} \psi(x, t) &= e^{ikx}\psi(t) \\ &= e^{ikx}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} \end{aligned} \tag{2.20}$$

From (2.19) and (2.10) we get

$$\begin{aligned} \left(\partial_t + \frac{\dot{a}}{2a} + \frac{ik}{a}\right)\psi_1 + im\psi_2 = 0 \\ \left(\partial_t + \frac{\dot{a}}{2a} - \frac{ik}{a}\right)\psi_2 + im\psi_1 = 0 \end{aligned} \tag{2.21}$$

Putting

$$\psi_1 = a^{-1/2}h_1 \tag{2.22}$$

and eliminating ψ_2 from (2.21), we get

$$\left(\partial_t^2 + \frac{k^2}{a^2} - \frac{ik\dot{a}}{av} + m^2\right)h_1 = 0 \tag{2.23}$$

We consider the following cases:

$$a(t) = a_0 t^{\pm 2/D} \tag{2.24}$$

where D denotes the total dimension. The scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$ is given by

$$R = \frac{2\ddot{a}}{a}$$

so that for $a = a_0 t^{-1}$, $R = 4t^{-2}$, and for $a = a_0 t$, $R = 0$. It can be verified easily that $T^{\mu}_{\mu} = 0$, where the energy-momentum tensor is given by

$$T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Case (i)

$$a(t) = a_0 t^{-1} \tag{2.25}$$

This is the case of the rapidly contracting universe (see footnote 2). Equation (2.23) then reduces to

$$\left[\partial_t{}^2 + \frac{k^2}{a_0^2} t^2 + \left(m^2 + \frac{ik}{a_0}\right)\right] h_1 = 0 \tag{2.26}$$

Putting

$$t^2 = z, \qquad \alpha_1 = m^2 + \frac{ik}{a_0} \tag{2.27}$$

we have that equation (2.26) takes the form

$$\left(4z \frac{\partial^2}{\partial z^2} + 2 \frac{\partial}{\partial z} + \frac{k^2}{a_0^2} z + \alpha_1\right) h_1 = 0 \tag{2.28}$$

or, putting

$$z = \beta y \quad \text{and} \quad h_1 = z^{-1/4} \varphi_1(z) \tag{2.29}$$

where $\beta = ia_0/k$, we have that equation (2.28) becomes

$$\left(\frac{\partial^2}{\partial y^2} + \frac{3/16}{y^2} - \frac{1}{4} + \frac{ia_0\alpha_1/4k}{y}\right)\varphi_1 = 0 \tag{2.30}$$

Comparing equation (2.30) with the standard Whittaker differential equation

$$\left(\frac{\partial^2}{\partial y^2} - \frac{1}{4} + \frac{\kappa}{y} + \frac{1/4 - \mu^2}{y^2}\right) W = 0 \tag{2.31}$$

we find that φ_1 has a solution of the form

$$\varphi_1 = W_{\kappa,\mu}(y)$$

where

$$\kappa = \frac{ia_0\alpha_1}{4k} = \frac{ia_0 m^2}{4k} - \frac{1}{4} \tag{2.32}$$

$$\mu = \frac{1}{4} \tag{2.33}$$

or

$$\psi = a_0^{-1/2} W_{\kappa,\mu}\left(\frac{t^2}{\beta}\right) \tag{2.34}$$

$W_{\kappa,\mu}(z)$ is the usual Whittaker function.

Now,

$$\psi_2 = \frac{i}{m}\left(\partial_t + \frac{\dot{a}}{2a} + \frac{ik}{a}\right)\psi_1$$

$$= \frac{i}{m}\frac{a_0^{-1/2}}{t}\left(2y\frac{\partial}{\partial y} - y - \frac{1}{2}\right)W_{\kappa,\mu} \tag{2.35}$$

Making use of the following recurrence relations for Whittaker functions

$$W_{\kappa+1/2,\mu} - y^{1/2} W_{\kappa,\mu+1/2} + (\kappa+\mu) W_{\kappa-1/2,\mu} = 0 \tag{2.36}$$

$$yW'_{\kappa,\mu} = \left(\frac{1}{2}y - \kappa\right)W_{\kappa,\mu} - W_{\kappa+1,\mu} \tag{2.37}$$

$$(2\kappa - y)W_{\kappa,\mu} + W_{\kappa+1,\mu} = \left(\mu - \kappa + \frac{1}{2}\right)\left(\mu + \kappa - \frac{1}{2}\right)W\kappa_{-1,\mu} \tag{2.38}$$

we obtain from (2.35)

$$\psi_2 = \left(-\frac{2ia_0^{-1/2}}{m\beta^{1/2}}\right)W_{\kappa+1/2,\mu}\left(\frac{t^2}{\beta}\right) \tag{2.39}$$

For $k \to -k$,

$$\kappa \to \kappa' = \frac{ia_0 m^2}{4k} + \frac{1}{4} = \kappa + \frac{1}{2}$$

Thus, proceeding in a similar way, we obtain

$$\psi_1(-k, t) = a_0^{-1/2} W_{\kappa+1/2,\mu}\left(\frac{t^2}{\beta}\right)$$

$$\psi_2(-k, t) = \frac{2ia_0^{-1/2}}{m\beta^{1/2}}\left(\kappa + \frac{1}{4}\right)W_{\kappa,\mu}\left(\frac{t^2}{\beta}\right)$$

Thus the complete solution to ψ is

$$\psi(k, t) = N_k a_0^{-1/2} \begin{bmatrix} W_{\kappa,\mu}\left(\dfrac{t^2}{\beta}\right) \\ \left(-\dfrac{2i}{m\beta^{1/2}}\right) W_{\kappa+1/2,\mu}\left(\dfrac{t^2}{\beta}\right) \end{bmatrix} e^{ikx}$$

$$\psi(-k, t) = N_k a_0^{-1/2} \begin{bmatrix} W_{\kappa+1/2,\mu}\left(\dfrac{t^2}{\beta}\right) \\ \dfrac{2i}{m\beta^{1/2}}\left(\kappa + \dfrac{1}{4}\right) W_{\kappa,\mu}\left(\dfrac{t^2}{\beta}\right) \end{bmatrix} e^{-ikx} \tag{2.40}$$

where N_k is a normalization constant.

Case (ii)

$$a(t) = a_0 t$$

This model was considered (in 3 + 1 dimensions) by Schrödinger in 1932 (Schrödinger, 1939, 1940). In this case equation (2.23) takes the form

$$\left[\partial_t^2 + \left(\frac{k^2}{a_0^2} + \frac{ik}{a_0}\right)\frac{1}{t^2} + m^2\right] h_1 = 0 \tag{2.41}$$

Comparing equation (2.41) with the Bessel differential equation

$$w'' + \left(\lambda^2 - \frac{\nu^2 - 1/4}{z^2}\right) w = 0 \tag{2.42}$$

where $w \equiv z^{1/2} J_\nu(\lambda z)$, we get

$$\lambda = m$$
$$\nu = \frac{1}{2} + \frac{ik}{a_0} \tag{2.43}$$

Thus

$$h_1 = t^{1/2} J_\nu(mt) \tag{2.44}$$

$$\psi_1 = a_0^{-1/2} J_\nu(mt) \tag{2.45}$$

Making use of the recurrence relations for $J_\nu(z)$,

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) \tag{2.46}$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z) \tag{2.47}$$

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z) \tag{2.48}$$

$$J'_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z) \tag{2.49}$$

we obtain

$$\psi_2 = i a_0^{-1/2} J_{\nu-1}(mt) \tag{2.50}$$

For

$$\begin{aligned} &k \to -k \\ &\nu \to \nu' = \frac{1}{2} - \frac{ik}{a_0} = 1 - \nu \end{aligned} \tag{2.51}$$

proceeding in a similar way, we obtain

$$\begin{aligned} \psi_1(-k, t) &= a_0^{-1/2} J_{-\nu+1}(mt) \\ \psi_2(-k, t) &= i a_0^{-1/2} J_{-\nu}(mt) \end{aligned} \tag{2.52}$$

Thus the complete solution is

$$\begin{aligned} \psi(k, t) &= N_k a_0^{-1/2} \begin{pmatrix} J_\nu(y) \\ iJ_{\nu-1}(y) \end{pmatrix} e^{ikx} \\ \psi(-k, t) &= N_k a_0^{-1/2} \begin{pmatrix} J_{-\nu+1}(y) \\ iJ_{-\nu}(y) \end{pmatrix} e^{-ikx} \end{aligned} \tag{2.53}$$

where N_k is some normalization constant.

3. DIRAC EQUATION FOR CIGAR-SHAPED METRIC

Here we consider the metric (called "cigar metric" by Witten)

$$ds^2 = \tanh^2 x \, dt^2 - dx^2, \qquad x \geq 0 \tag{3.1}$$

The scalar curvature is given by

$$R = 4 \operatorname{sech}^2 x \tag{3.2}$$

which is regular at $x = 0$.

Calculation of the energy-momentum tensor

$$T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

easily shows that $T^\mu_\mu = 0$.

The metric (3.1) can be obtained from the Schwarzschild metric in the following way. The Schwarzschild metric is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

Making a change of coordinates

$$u = (u' + v')/2$$

$$v = (u' - v')/2$$

where

$$u' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh\left(\frac{t}{4M}\right)$$

$$v' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh\left(\frac{t}{4M}\right)$$

we find that the Schwarzschild metric takes the form

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r} du\, dv + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{3.3}$$

where u, v are Kruskal–Szekeres coordinates. Discarding the radial part and after a suitable coordinate transformation (Ellis *et al.*, 1992), we can write the metric (3.3) as a conformally rescaled form of the two-dimensional black hole metric

$$ds^2 = \frac{1}{1 - uv} du\, dv \tag{3.4}$$

which is singular at $uv = 1$. Putting

$$2u = -e^{x' - t}$$

$$2v = e^{x' + t}$$

where

$$x' = x + \ln(1 - e^{-2x})$$

it is easily seen that equation (3.4) takes the form of equation (3.1). The physical singularity of (3.4) is at $uv = 1$, where the curvature blows up, and consists of a past and future branch. The past branch is a naked singularity. The future branch is the black hole singularity, from which no signal can cross the horizon to an observer in the original x, t region. For our metric, the dyads are

$$e_0^0 = \coth x, \qquad e_1^1 = 1, \qquad e_0^1 = e_1^0 = 0 \tag{3.5}$$

and the Dirac equation takes the form

$$\left[iw\coth x+\gamma^0\gamma^1\left(\partial_x+\frac{1}{\sinh 2x}\right)+im\gamma^0\right]\psi=0 \tag{3.6}$$

where

$$\begin{aligned}\psi(x,t)&=e^{-iwt}\psi(x)\\&=e^{-iwt}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix}\end{aligned} \tag{3.7}$$

or,

$$\left(-iw\coth x+\partial_x+\frac{1}{\sinh 2x}\right)\psi_1+im\psi_2=0 \tag{3.8}$$

$$\left(-iw\coth x-\partial_x-\frac{1}{\sinh 2x}\right)\psi_2+im\psi_1=0 \tag{3.9}$$

Putting

$$\psi_2=\tanh^{-1/2}x\ \varphi_2(x)$$

and eliminating ψ_1 from (3.8), we obtain

$$[\partial_x^2+(w^2-iw)\operatorname{cosech}^2 x+(w^2-m^2)]\varphi_2=0 \tag{3.10}$$

Putting $y=-\sinh^2 x$, we see that (3.10) reduces to

$$y(1-y)\frac{\partial^2\varphi_2}{dy^2}+\left(\frac{1}{2}-y\right)\frac{\partial\varphi_2}{\partial y}-\left[\frac{k^2}{4}-\frac{\beta(\beta-1)}{4y}\right]\varphi_2=0 \tag{3.11}$$

where $\beta=-iw$. The general solution of (3.11) can be written as

$$\varphi_2=y^{\beta/2}\left[AF\left(a,b,\frac{1}{2},1-y\right)+B(1-y)^{1/2}F\left(a+\frac{1}{2},b+\frac{1}{2},\frac{3}{2},1-y\right)\right] \tag{3.12}$$

Then ψ can be written as

$$\psi=e^{-iwt}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{3.13}$$

Expressions for ψ_1,ψ_2 which are consistent with the flat-space limit are given by

$$\psi_1=N_w z^{1/2}(z-1)^{1/4-iw/2}F\left(\frac{1}{2}-a,\frac{1}{2}-b,\frac{1}{2},z\right) \tag{3.14}$$

$$\psi_2=\bar{N}_w z^{1/2}(z-1)^{1/4+iw/2}F\left(1+a,1+b,\frac{3}{2},z\right) \tag{3.15}$$

where

$$z = \cosh^2 x, \qquad a = \frac{i}{2}(w + k), \qquad b = \frac{i}{2}(w - k) \tag{3.16}$$

To show that (3.14) and (3.15) have the correct asymptotic behavior (i.e., for large x, we should get the flat-space behavior) we use the asymptotic formula

$$F(a, b, c, z) \sim \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)}(-z)^{-a} \tag{3.17}$$

It can be easily seen from (3.14) and (3.15) that for large x, we have $\psi_1, \psi_2 \sim e^{\pm ikx}$. Here N_w and $\bar{N}_w$ are normalization constants related by

$$|N_w| = \frac{4|\bar{N}_w|ab}{m}$$

4. DISCUSSION AND CONCLUSION

In this paper we have considered the Dirac equation in (1 + 1)-dimensional curved space-time. We have considered two kinds of metric, a flat Robertson–Walker space and the so-called cigarlike metric obtained by Witten. Now, to study quantum field theory in curved space-time, one must quantize the solutions obtained. To obtain canonically quantized modes, one must get a complete set of orthonormal wave functions with an inner product defined over a Cauchy-like surface Σ:

$$\langle \chi | \varphi \rangle = \int_\Sigma d\Sigma\, \bar{\chi} e^a_\mu \gamma_a \varphi \tag{4.1}$$

This essentially implies the evaluation of the integrals of the product of the normalized wave functions.

In general the evaluation of these integrals in closed form is difficult. However, in asymptotic cases the integrals can often be calculated without much difficulty. Quantization of the wave functions and its application will be taken up in a future publication.

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